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Imprints of a Primordial Magnetic Field upon the Cosmic Microwave Background Anisotropy and Polarization

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We review the imprints that a primordial magnetic field may have left upon the cosmic microwave background (CMB) anisotropy and polarization through Faraday rotation around the time of decoupling. Differential Faraday rotation reduces the degree of linear polarization acquired through anisotropic Thomson scattering. Depolarization reduces the damping due to photon diffusion, which results in an increase of the anisotropy on small angular scales. The effect is significant at frequencies around and below 10 GHz $\cdot \sqrt{B_0/10^{-9}}$ G where B_0 is the present strength of the primordial magnetic field.

1. INTRODUCTION

Present and future cosmic microwave background measurements, particularly those planned through a new generation of satellite experiments such as MAP[2] and Planck Surveyor (COBRAS/SAMBA),[3] offer the perspective of a very accurate determination of the angular power spectrum of its temperature anisotropies, all the way down to very small scales. It is plausible that these measurements will serve to accurately determine several cosmological parameters (Jungman *et al.*, 1996a, b; Bond *et al.*, 1997; Zaldarriaga *et al.*, 1997), such as the total and baryonic mass densities and the Hubble constant, unless they take values in regions of parameter space of large degeneracy. Measurement of the (as yet undetected) polarization of the CMB may significantly help to break eventual degeneracies.

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[2] See the MAP home page at http://map.gsfc.nasa.gov.
[3] See the COBRAS/SAMBA homepage at http://astro.estec.esa.nl:80/SA-general/Projects/Cobras/cobras.html.

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The existence of a hypothetical magnetic field of primordial origin, and the determination of its strength, is another task that cosmic microwave background measurements may have the potential to accomplish. In this article we shall review some of the effects that a magnetic field may have upon the anisotropy and polarization properties of the CMB. A very distinctive effect is its ability to depolarize the CMB through differential Faraday rotation across the last scattering surface (Harari *et al.*, 1997), and this is what we shall discuss in more depth. We shall see that the effect is significant, albeit only at frequencies relatively low for an efficient detection.

2. PRIMORDIAL MAGNETIC FIELDS AND THE CMB

It is conceivable that the large-scale magnetic fields observed in galaxies and clusters have their origin in a primordial field, which present observations constrain to have a strength below $B_0 \approx 10^{-9}$ G (Asseo and Sol, 1987; Kronberg, 1994). Since a primordial magnetic field scales as $B(t) = B(t_0)a^2(t_0)/a^2(t)$, where $a(t)$ is the Robertson–Walker scale factor, it should have had a strength below $B_* \approx 10^{-3}$ G at a redshift $z_* = 1000$, around the time when the CMB photons we observe today last-scattered from free electrons. Direct bounds on the strength of a hypothetical magnetic field at those early times are not very stringent. Compatibility with big-bang nucleosynthesis places an upper bound that, extrapolated to the time of recombination, is at most $B_* = 0.1$ G (Cheng *et al.*, 1996; Grasso and Rubinstein, 1996).

There are several mechanisms through which a primordial magnetic field may leave its imprint upon the CMB. A large-scale, homogeneous magnetic field could drive an anisotropic expansion of the universe, and the anisotropic redshift would manifest itself as a large-scale temperature anisotropy (Zeldovich and Novikov, 1983). Recent upper limits, derived from a statistical analysis of the 4-year COBE data for anisotropy, suggest $B_0 < 3.4 \times 10^{-9} (\Omega_0 h_{50}^2)^{1/2}$ G (Barrow *et al.*, 1997).

Another potential imprint of a primordial magnetic field upon the CMB arises from photon–graviton conversion. In an external, homogeneous magnetic field B, photons and gravitons can convert into each other conserving energy and linear momentum with a probability given by (Gerstsenshtein, 1962; Zeldovich and Novikov, 1983; Cillis and Harari, 1996) $P = 4\pi G B^2 L^2 \sin^2\theta \approx 8 \times 10^{-50} (B/G)(L/\text{cm})^2 \sin^2\theta$. Here G is Newton's constant (while G denotes gauss), L is the distance covered by the photon (or graviton) in the magnetic field, and θ is the angle between the external magnetic field and the common direction of propagation of photons and gravitons. The angular dependence of the conversion process in a primordial magnetic field has the potential to induce CMB anisotropies (Zeldovich and Novikov, 1983;

Magueijo, 1994; Chen, 1995). The effect would be significant in the absence of free electrons. Plasma effects, however, make the characteristic length for photon–graviton oscillations much shorter than the Hubble radius, preventing the conversion probability from accumulating significantly over cosmological scales, and thus making the effect negligible (Cillis and Harari, 1996). Indeed, in the presence of a free electron density n_e, photons propagate as if they had an effective mass equal to the plasma frequency $\omega_{pl} = (4\pi\alpha n_e/m_e)^{1/2}$, where m_e denotes the electron mass and $\alpha = e^2/4\pi \sim 1/137$ is the fine structure constant. The conversion probability becomes very sensitive to the coherence of the photon–graviton oscillations, and is at most of order $P = GB^2L\ell_{osc} \sin^2 \theta$, where $\ell_{osc} = 4\pi\omega/\omega_{pl}^2$ is the oscillation length and ω the photon frequency. A primordial magnetic field of present value around 10^{-9} G induces a photon–graviton conversion rate that has negligible effects upon the CMB anisotropy.

It is more speculative, but in principle possible, that photons also convert into hypothetical pseudoscalar particles, with electromagnetic couplings such as that of the axion (Sikivie, 1983; Raffelt and Stodolsky, 1988). If the pseudoscalar particle is massless or extremely light, and the pseudoscalar–electromagnetic coupling were sufficiently strong, conversion in a primordial field could induce not only anisotropies but also linear polarization on the CMB (Harari and Sikivie, 1992).

A primordial magnetic field may also distort the CMB anisotropy power spectrum on small angular scales due to its impact upon the photon–baryon fluid sound speed, as was recently discussed in Adams *et al.*, (1996).

3. FARADAY ROTATION OF THE CMB POLARIZATION

A distinctive feature of a primordial magnetic field is its ability to affect the polarization properties of the CMB through the effect of Faraday rotation (Milaneschi and Fabbri, 1985; Kosowsky and Loeb, 1996; Harari *et al.*, 1997). The CMB is expected to have acquired a small degree of linear polarization through Thomson scattering (Rees, 1968), which polarizes the radiation if there is a quadrupole anisotropy in its distribution function. Typically, the CMB degree of linear polarization is expected to be more than ten times smaller than the relative temperature anisotropy on comparable angular scales (Bond and Efstathiou, 1987; Zaldarriaga and Harari, 1995). An early reionization of the universe after recombination may have yielded a larger degree of polarization (Zaldarriaga, 1997). The CMB has not yet been observed to be polarized, the upper limit on its degree of linear polarization on large angular scales being $P < 6 \times 10^{-5}$ (Lubin and Smoot, 1983). In this and following sections we shall review the imprints that Faraday rotation may leave upon the CMB anisotropy and polarization.

After traversing a distance L in a direction $\hat{q}$ within a homogeneous magnetic field B, linearly polarized radiation has its plane of polarization rotated an angle

$$\varphi = \frac{e^3 n_e x_e \mathbf{B} \cdot \hat{q}}{8\pi^2 m^2 c^2} \lambda^2 L \tag{1}$$

n_e is the total number-density of electrons and x_e its ionized fraction. λ is the wavelength of the radiation, m is the electron mass, and c is the speed of light. We work in Heaviside–Lorentz electromagnetic units ($\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant if we take $\hbar = c = 1$).

The effect of Faraday rotation upon the CMB can be traced in the Boltzmann equations for the temperature and polarization fluctuations. We follow the notation and formalism of Zaldarriaga and Harari (1995). The total temperature fluctuation is denoted by Δ_T, while the fluctuations in the Stokes parameters Q and U are denoted by Δ_Q and Δ_U, respectively. The degree of linear polarization is given by $\Delta_P = (\Delta_Q^2 + \Delta_U^2)^{1/2}$. All three quantities are expanded in Legendre polynomials as $\Delta_X = \Sigma_l (2l + 1)\Delta_{Xl} P_l(\mu)$, where $\mu = \cos\theta = \mathbf{k}\ \hat{q}/|\mathbf{k}|$ is the cosine of the angle between the wave vector of a given Fourier mode $\mathbf{k}$ and the direction of photon propagation $\hat{q}$. The evolution equations for the Fourier mode of wave vector $\mathbf{k}$ read

$$\dot{\Delta}_T + ik\mu(\Delta_T + \Psi) = -\dot{\Phi} - \dot{\kappa}\left[\Delta_T - \Delta_{T_0} - \mu V_b + \frac{1}{2} P_2(\mu) S_P\right] \tag{2}$$

$$\dot{\Delta}_Q + ik\mu\Delta_Q = -\dot{\kappa}\left[\Delta_Q - \frac{1}{2}(1 - P_2(\mu))S_P\right] + 2\omega_B \Delta_U \tag{3}$$

$$\dot{\Delta}_U + ik\mu\Delta_U = -\dot{\kappa}\Delta_U - 2\omega_B \Delta_Q \tag{4}$$

We have defined

$$S_P \equiv -\Delta_{T_2} - \Delta_{Q2} + \Delta_{Q0} \tag{5}$$

which acts as the effective source term for the polarization. V_b is the bulk velocity of the baryons, which satisfies the continuity equation

$$\dot{V}_b = -\frac{\dot{a}}{a} V_b - ik\Psi + \frac{\dot{\kappa}}{R}(3\Delta_{T_1} - V_b) \tag{6}$$

An overdot means derivative with respect to the conformal time $\tau = \int dt\ a_0/a$, with $a(t)$ the scale factor of the spatially flat Robertson–Walker metric, and $a_0 = a(t_0)$ its value at present. We consider scalar (energy-density) metric fluctuations, which we describe in terms of the gauge-invariant gravitational potentials Ψ and Φ. $R \equiv 3\rho_b/4\rho_\gamma$ coincides with the scale factor $a(t)$ normalized to 3/4 at the time of equal baryon and radiation densities. $\kappa = x_e n_e \sigma_T a/a_0$ is the Thomson scattering rate, or differential optical depth, with n_e the electron number density, x_e its ionized fraction, and σ_T the Thomson scattering cross section. Finally, ω_B is the Faraday rotation rate (Kosowsky and Loeb, 1996).

$$\omega_B \equiv \frac{d\varphi}{d\tau} = \frac{e^3 n_e x_e \mathbf{B}\,\hat{q}}{8\pi^2 m^2 \nu^2}\frac{a}{a_0} \tag{7}$$

Equation (2) shows that density inhomogeneities, described by the gravitational potentials Φ and Ψ, are a source of temperature anisotropies. Equation (3) displays how Thomson scattering in the presence of anisotropies generates polarization. If there were no magnetic field, one could choose a basis for the Stokes parameters such that $U = 0$. This is not so when Faraday rotation by a magnetic field breaks the axial symmetry around $\mathbf{k}$ and mixes Q and U, as is displayed by (3) and (4).

Equations (2)–(4) can be solved in the tight-coupling approximation, which amounts to an expansion in powers of $k\tau_C$, where $\tau_C \equiv \dot{\kappa}^{-1}$ is the average conformal time between collisions. To first order in $k\tau_C$ the tight-coupling solutions in the presence of a homogeneous magnetic field, which we take for simplicity such that $\mathbf{B} \parallel \mathbf{k}$, are such that

$$\Delta_U = -F\cos\theta\,\Delta_Q; \qquad \Delta_Q = \frac{3}{4}\frac{S_P \sin^2\theta}{4(1 + F^2\cos^2\theta)} \tag{8}$$

and

$$S_P = \frac{4}{3(3-2d)}\, ik\tau_C \Delta_{T_1} = -\frac{4}{3(3-2d)}\,\tau_C \dot{\Delta}_0 \tag{9}$$

where we have defined the coefficient F as

$$F\cos\theta \equiv 2\omega_B \tau_C \tag{10}$$

and so

$$F \doteq \frac{e^3}{4\pi^2 m^2 \sigma_T}\frac{B}{\nu^2} \approx 0.7\left(\frac{B_*}{10^{-3}\ \text{gauss}}\right)\left(\frac{10\ \text{GHz}}{\nu_0}\right)^2 \tag{11}$$

The coefficient F represents the average Faraday rotation between collisions. The coefficient d is defined as

$$d \equiv \frac{15}{8}\left[\frac{\arctan(F)}{F}\left(1 + \frac{2}{F^2} + \frac{1}{F^4}\right) - \frac{5}{3F^2} - \frac{1}{F^4}\right] \tag{12}$$

Our analytic expressions are valid for the special case $\mathbf{B} \parallel \mathbf{k}$. In a more general situation they are likely to be more complicated. Nevertheless, at the end we are interested in the stochastic superposition of all Fourier modes, with a Gaussian spectrum with no privileged direction. Average quantities may depend on the angle between the magnetic field direction and the line of sight, but not upon the angle between **B** and a particular **k**. The simplification made by considering **B** || **k** at most underestimates the average depolarizing effect of the magnetic field, since it corresponds to the situation in which the magnetic field is perpendicular to the direction of maximum polarization.

Notice that in the tight-coupling approximation all quantities of interest can be expressed in terms of the monopole Δ_{T_0}, which in turn satisfies the equation of a forced and damped oscillator. In terms of the quantity $\Delta_0 \equiv \Delta_{T_0} + \Phi$, and neglecting $O(R^2)$ contributions, the equation for Δ_0 reads

$$\ddot{\Delta}_0 + \left[\frac{\dot{R}}{1 + R} + \frac{16}{90}\frac{(5 - 3d)}{(3 - 2d)}\frac{k^2\tau_C}{(1 + R)}\right]\dot{\Delta}_0 + \frac{k^2}{3(1 + R)}\Delta_0 = \frac{k^2}{3(1 + R)}[\Phi - (1 + R)\Psi] \quad (13)$$

The damping term in this equation depends (through d) upon the depolarizing effect of Faraday rotation, and is reduced by a factor 5/6 at frequencies such that $d << 1$, for which depolarization is significant.

4. DEPOLARIZATION BY A PRIMORDIAL FIELD

The polarization and anisotropy observed at present times can be found analytically with high accuracy through integration of the tight-coupling solutions across the width of the last scattering surface. We refer the reader to Zaldarriaga and Harari (1995) and Hu and Sugiyama (1995a, b) for more details. Still, the basics of the depolarizing effect of a primordial magnetic field can be read off from equations (8) and (9). When there is no magnetic field ($F = 0$, $d = 1$) $\Delta_U = 0$ and $\Delta_Q = (15/8)\Delta_{T_2} \sin^2\theta$. A magnetic field generates Δ_U, through Faraday rotation and reduces Δ_Q. In the limit of very large F (large Faraday rotation between collisions) the polarization vanishes. In other words, photons that scatter at different times suffer diverse amounts of Faraday rotation, the net effect resulting in depolarization.

The net depolarizing factor depends upon the angle between the line of sight and the orientation of the magnetic field at the time of decoupling in the region under observation. On average over many regions separated by more than a few degrees (the angle subtended by the Hubble radius at decoupling) we estimate, after integration across the last scattering surface, a depolarizing factor of order

$$\overline{D} = \frac{1}{\sqrt{1 + F^2/2}} f(F) \quad (14)$$

where

$$f(F) \equiv \frac{7}{1+6d}\left[1-\frac{\ln(3-2d)}{\ln(10/3)}\right] \tag{15}$$

At low frequencies, those for which the effect is large, the average depolarizing factor scales as

$$\bar{D} \approx 0.6\,\frac{\sqrt{2}}{F} \approx 0.85\left(\frac{\nu_0}{\nu_d}\right)^2 \qquad \text{if} \quad \nu_0 << \nu_d \tag{16}$$

where we denoted by ν_d the frequency at which depolarization starts to be significantly large, defined through $F \equiv (\nu_d/\nu_0)^2$, and so

$$\nu_d \approx 8.4 \text{ GHz} \cdot \left(\frac{B_*}{10^{-3}\text{ G}}\right)^{1/2} \tag{17}$$

The polarization properties of the CMB at sufficiently low frequencies are thus very sensitive to a hypothetical primordial magnetic field.

5. EFFECTS UPON THE CMB ANISOTROPY

The imprints of Faraday rotation in a primordial magnetic field can also be looked for in the anisotropy of the CMB on small angular scales. Indeed, the polarization properties of the CMB feed back into its anisotropy. In particular, they affect photon diffusion (Kaiser, 1983; Hu *et al.*, 1995), which damps anisotropies on small angular scales (Silk, 1968; Peebles, 1980). The effect can be read off from the damping term in (13). Solving the tight-coupling equations to second order, one finds that

$$\Delta_T(\tau) = \Delta_T e^{i\omega_0\tau} e^{-\gamma\tau} \tag{18}$$

where $\omega_0 = k/\sqrt{3(1+R)}$ and

$$\gamma(d) = \frac{k^2\tau_C}{6(1+R)}\left(\frac{8}{15}\,\frac{(5-3d)}{15(3-2d)} + \frac{R^2}{1+R}\right) \tag{19}$$

The photon-diffusion damping length scale depends upon the primordial magnetic field, due to the depolarizing effect of Faraday rotation, through the coefficient d. This has a significant impact upon the anisotropy of the CMB on small angular scales, at frequencies where the depolarizing effect is large. It is possible to make an accurate analytic estimate of the effect upon the multipole coefficients of the CMB temperature anisotropy correlation function:

$$C(\theta) = \langle \Delta_T(\hat{n}_1)\Delta_T(\hat{n}_2)\rangle_{\hat{n}_1\cdot\hat{n}_2=\cos\theta} = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_lP_l(\cos\theta) \quad (20)$$

The relative change of the C_l in a standard cold dark matter cosmological model as function of the depolarizing coefficient d reads (Zaldarriaga and Harari, 1995)

$$\Delta C_l = \frac{C_l(d)}{C_l(d=1)} - 1 \approx \exp\left(\frac{(l/1500)^2(1-d)}{6-4d}\right) - 1 \quad (21)$$

This analytic estimate follows very accurately the numerical solution obtained with the addition of the Faraday rotation term to the code CMBFast (Seljak and Zaldarriaga, 1996a). In Fig. 1 we plot ΔC_l (expressed as a percentage) at $l = 1000$ as a function of frequency for three different values of the magnetic field B_*. The effect is larger at smaller angular scales (larger l). We display it at $l = 1000$, which is expected to be accessible by the next CMB satellite experiments MAP and Planck Surveyor.

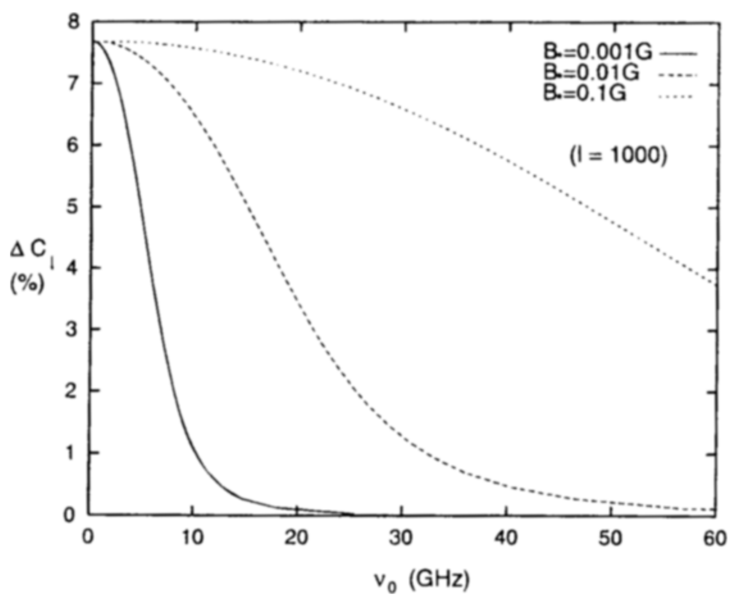

Fig. 1. Percentual change due to depolarization by Faraday rotation of the $l = 1000$ anistropy correlation function multipoles as a function of the CMB frequency for different values of the primordial magnetic field around decoupling.

6. ADDITIONAL EFFECTS OF FARADAY ROTATION

Depolarization by Faraday rotation is a very distinctive signature of a primordial magnetic field. Other footprints of a primordial field upon the CMB polarization may be looked for in the polarization pattern in the sky, which is affected by Faraday rotation (Kosowsky and Loeb, 1996).

CMB polarization measurements will be crucial to help determine cosmological parameters that CMB anisotropy measurements alone may not unambiguously fix (Zaldarriaga *et al.*, 1997). For instance, the tensor to scalar ratio of primordial fluctuations is not easy to determine with CMB anisotropy measurements alone, and CMB polarization measurements may serve to fix it (Harari and Zaldarriaga, 1993). It was recently shown that there is a particular combination of Stokes parameters Q and U that vanishes if the CMB polarization is due to scalar fluctuations, while it is nonzero if it arises as a consequence of tensor fluctuations (gravitational waves) (Zaldarriaga and Seljak, 1997; Seljak and Zaldarriaga, 1996b; Kamionkowski *et al.*, 1996). Faraday rotation is likely to change this conclusion, since even for scalar fluctuations alone it mixes Q and U in such a way that the otherwise vanishing combination of Stokes parameters becomes proportional to the other, nonvanishing independent combination. However, since the effect is roughly proportional to our parameter F, it will only affect relatively low frequencies.

7. CONCLUSIONS

A primordial magnetic field leaves significant imprints upon the CMB through the effect of Faraday rotation of its polarization. A distinctive feature is depolarization: the degree of linear polarization of the CMB is significantly reduced at those frequencies for which the average Faraday rotation during collisions just before decoupling was large. The effect is characterized by the parameter $F \approx 0.7(B_0/10^{-9}\ \mathrm{G})(10\ \mathrm{GHz}/\nu_0)^2$.

Faraday rotation affects not only the polarization properties of the CMB, but also its anisotropy. Depolarization reduces the viscous damping of anisotropies due to photon diffusion, which results in a significant increase of the anisotropy on small angular scales, albeit only at rather low frequencies, around or below $10\ \mathrm{GHz} \cdot \sqrt{B_0/10^{-9}\ \mathrm{G}}$. Measurements of the anisotropy and polarization of the CMB at sufficiently low frequencies may thus be able to trace the footprints of a hypothetical primordial magnetic field around the time of decoupling.

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